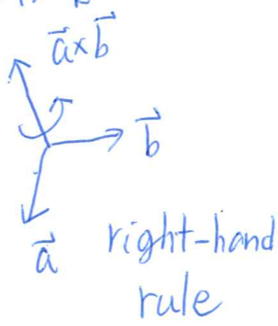
Cross Product

Let $\vec{a}, \vec{b} \in \mathbb{R}^3$. Define

$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2)\hat{i} + (-a_1 b_3 + a_3 b_1)\hat{j} + (a_1 b_2 - a_2 b_1)\hat{k}$$

$$= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$



Properties ① $i \times j = k$, $j \times k = i$, $k \times i = j$,

② $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$,

③ $\vec{a} \times \vec{a} = 0$,

④ $\vec{a} \times (\alpha \vec{b} + \beta \vec{c}) = \alpha \vec{a} \times \vec{b} + \beta \vec{a} \times \vec{c}$

⑤ $(\alpha \vec{a}) \times \vec{b} = \vec{a} \times (\alpha \vec{b}) = \alpha (\vec{a} \times \vec{b})$.

Dot product & cross product

~ $|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$ (a direct check)

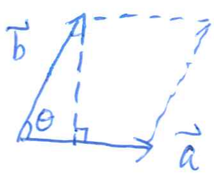
~ $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\,|\sin\theta|$, θ angle bet. $\vec{a}$ & $\vec{b}$.

(Pf: $|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 - |a|^2 |b|^2 \cos^2\theta$
$= |\vec{a}|^2 |b|^2 \sin^2\theta$.)

~ $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$

(a direct check)

~ $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{0}$ (Jacobi's identity)



geometric meaning

~ $|\vec{a} \times \vec{b}|$ is the area of of the parallelogram formed by $\vec{a}, \vec{b}$

~ $|\vec{a} \cdot (\vec{b} \times \vec{c})|$ is the volume of the parallelepiped formed by $\vec{a}, \vec{b}, \vec{c}$.

(draw it yourself)

$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$

$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

Q1: i) $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$

ii) $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = 0$

iii) $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c}) = \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$

Ans: i) $\vec{a}\times(\vec{b}\times\vec{c}) = (a_1\vec{i} + a_2\vec{j} + a_3\vec{k})\times\left[(b_2c_3 - b_3c_2)\vec{i} - (b_1c_3 - b_3c_1)\vec{j} + (b_1c_2 - b_2c_1)\vec{k}\right]$

$$= \left[a_2(b_1c_2 - b_2c_1) - a_3(b_1c_3 - b_3c_1)\right]\vec{i}$$

$$- \left[a_1(b_1c_2 - b_2c_1) - a_3(b_2c_3 - b_3c_2)\right]\vec{j}$$

$$+ \left[a_1(b_1c_3 - b_3c_1) - a_2(b_2c_3 - b_3c_2)\right]\vec{k}$$

$\vec{b}(\vec{a}\cdot\vec{c}) = (b_1\vec{i} + b_2\vec{j} + b_3\vec{k})(a_1c_1 + a_2c_2 + a_3c_3)$

$\vec{c}(\vec{a}\cdot\vec{b}) = (c_1\vec{i} + c_2\vec{j} + c_3\vec{k})(a_1b_1 + a_2b_2 + a_3b_3)$

ii)
$$\vec{a}\times(\vec{b}\times\vec{c}) = \cancel{\vec{b}(\vec{a}\cdot\vec{c})} - \cancel{\vec{c}(\vec{a}\cdot\vec{b})}$$
$$\vec{b}\times(\vec{c}\times\vec{a}) = \cancel{\vec{c}(\vec{b}\cdot\vec{a})} - \cancel{\vec{a}(\vec{b}\cdot\vec{c})}$$
$$+\quad \vec{c}\times(\vec{a}\times\vec{b}) = \cancel{\vec{a}(\vec{c}\cdot\vec{b})} - \cancel{\vec{b}(\vec{c}\cdot\vec{a})}$$

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iii) $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \vec{a}\cdot(\vec{b}\times(\vec{c}\times\vec{d}))$

$$= \vec{a}\cdot\left(\vec{c}(\vec{b}\cdot\vec{d}) - \vec{d}(\vec{b}\cdot\vec{c})\right)$$

$$= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$$

Line integral of vector fields on $\mathbb{R}^2$

$\vec{F} = (P, Q)$

$\int_C \vec{F}\cdot\vec{T}\, ds = \int P\,dx + Q\,dy$

$\int_C \vec{F}\cdot\vec{n}\, ds = \int -Q\,dx + P\,dy$

Green's thm: $\int_C \vec{F}\cdot\vec{T}\, ds = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$

$\int_C \vec{F}\cdot\vec{n}\, ds = \iint_D \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\, dA$

Q2: Show that i) $\int_C \nabla f\cdot\vec{n}\, ds = \iint_D \Delta f\, dA$

ii) If $\Delta f \equiv 0$, then the average of f over any circle centered at $(0,0)$ is equal to $f(0,0)$

(Hint: Show that $I(r) = \frac{1}{2\pi r}\int_{C_r} f$ is independent of r, where C_r is the circle centered at origin with radius r)

Ans: We recall $\Delta f = f_{xx} + f_{yy}$

i) L.H.S: $\int_C \nabla f\cdot\vec{n} = \int_C -f_y\,dx + f_x\,dy$

$= \iint_D f_{xx} + f_{yy}\, dA$

$= \iint_D \Delta f\, dA$

ii) Parametrization of C_r: $\gamma(t) = (r\cos t, r\sin t)$ $\quad 0 \le t \le 2\pi$

$\gamma'(t) = (-r\sin t, r\cos t)$ $\quad |\gamma'(t)| = r$

$dx = -r\sin t\, dt$ $\quad dy = r\cos t\, dt$

$I(r) = \frac{1}{2\pi r}\int_0^{2\pi} f(r\cos t, r\sin t)\cdot r\, dt$

$= \frac{1}{2\pi}\int_0^{2\pi} f(r\cos t, r\sin t)\, dt$

$\frac{d}{dr} I(r) = \frac{1}{2\pi}\int_0^{2\pi} f_x\cos t + f_y \sin t\, dt$

$= \frac{1}{2\pi r}\int_0^{2\pi} -f_x\,dy + f_y\,dx$

$= -\frac{1}{2\pi r}\iint_D f_{xx} + f_{yy} = 0$

Thus $I(r) = \lim_{\rho\to 0} I(\rho) = f(0,0)$ by the continuity of f.

Green's identities

1) $\iint_D f\Delta g + \nabla f \cdot \nabla g \, dA = \int_C f\nabla g \cdot \vec{n} \, ds$

2) $\iint_D f\Delta g - g\Delta f = \int_C (f\nabla g - g\nabla f)\cdot \vec{n} \, ds$

Pf: 1): $\int_C f\nabla g \cdot \vec{n} \, ds = \int_C (fg_x, fg_y)\cdot \vec{n} \, ds$

$$= \iint_D (fg_x)_x + (fg_y)_y \, dA$$

$$= \iint_D f_x g_x + f g_{xx} + f_y g_y + f g_{yy} \, dA$$

$$= \iint_D f(g_{xx} + g_{yy}) + f_x g_x + f_y g_y \, dA$$

$$= \iint_D f\Delta g + \nabla f \cdot \nabla g \, dA$$

2) $\iint_D f\Delta g + \nabla f \cdot \nabla g \, dA = \int_C f\nabla g \cdot n \, ds$

$-)$ $\iint_D g\Delta f + \nabla g \cdot \nabla f \, dA = \int_C g\nabla f \cdot n \, ds$

$\iint_D f\Delta g - g\Delta f \, dA = \int_C (f\nabla g - g\nabla f)\cdot n \, ds$

Rmk: Some special cases of 1):

when $f = 1 \Rightarrow \iint_D \Delta g \, dA = \int_C \nabla g \cdot n \, ds$

when $f = g \Rightarrow \iint_D f\Delta f + \|\nabla f\|^2 \, dA = \int_C f\nabla f \cdot n \, ds$

Application:

Suppose $\Delta f = \Delta g$ on D

and $f = g$ on C

then $f = g$ on D.

pf: Let $h = f - g$, then $\Delta h = 0$ on D and $h = 0$ on C.

we need to show $h = 0$ on D.

Using the identity

$$\iint_D h \cancel{\Delta h} + \|\nabla h\|^2 \, dA = \int_C \cancel{h} \nabla h \cdot n \, ds$$

$$\Rightarrow \iint_D \|\nabla h\|^2 \, dA = 0$$

$$\Rightarrow \nabla h \equiv 0$$

$$\Rightarrow h \text{ constant}$$

but $h = 0$ on C, so $h = 0$ on D.

$\underline{Q3}$ $\vec{r} = \vec{r}(u,v)$ be a parametrization of a surface S in $\mathbb{R}^3$, $(u,v) \in R$

Let $E(u,v) = \|r_u\|^2$

$F(u,v) = r_u \cdot r_v$

$G(u,v) = \|r_v\|^2$

Show that $\text{Area}(S) = \iint_R \sqrt{EG - F^2}\, du dv$

Ans: $\text{Area}(S) = \iint_R \|r_u \times r_v\|\, du dv$

It suffices to show $\|r_u \times r_v\|^2 = EG - F^2$

In other words, showing

$$\|r_u \times r_v\|^2 = \|r_u\|^2 \|r_v\|^2 - (r_u \cdot r_v)^2$$

$$\Leftrightarrow \|r_u\|^2 \|r_v\|^2 \sin^2\theta = \|r_u\|^2 \|r_v\|^2 - \|r_u\|^2 \|r_v\|^2 \cos^2\theta$$

(θ is the angle between r_u and r_v)

which is true by the identity $\sin^2\theta + \cos^2\theta = 1$

Q4 Find the surface area of the torus

$$\vec{r}(\theta,\varphi) = \Big((a\cos\theta + b)\cos\phi,\ (a\cos\theta+b)\sin\phi,\ a\sin\theta \Big) \qquad \theta, \phi \in [0, 2\pi]$$

Ans: $\vec{r}_\theta = (-a\sin\theta\cos\phi,\ -a\sin\theta\sin\phi,\ a\cos\theta)$

$\vec{r}_\phi = (-(a\cos\theta+b)\sin\phi,\ (a\cos\theta+b)\cos\phi,\ 0)$

$$E = \|\vec{r}_\theta\|^2 = a^2\sin^2\theta\cos^2\phi + a^2\sin^2\theta\sin^2\phi + a^2\cos^2\theta$$
$$= a^2\sin^2\theta + a^2\cos^2\theta$$
$$= a^2$$

$$F = \vec{r}_\theta \cdot \vec{r}_\phi = a\sin\theta(a\cos\theta+b)\sin\phi\cos\phi - a\sin\theta(a\cos\theta+b)\sin\phi\cos\phi = 0$$

$$G = \|r_\phi\|^2 = (a\cos\theta+b)^2\sin^2\phi + (a\cos\theta+b)^2\cos^2\phi$$
$$= (a\cos\theta+b)^2$$

$$\sqrt{EF-G^2} = a(a\cos\theta+b) > 0$$

Now surface area $= \int_0^{2\pi}\int_0^{2\pi} a(a\cos\theta+b)\,d\theta\,d\phi$

$$= \int_0^{2\pi} 2\pi ab\,d\phi$$

$$= 4\pi^2 ab$$